Satisfiability-based algorithms for Boolean optimization

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This paper proposes new algorithms for the Binate Covering Problem (BCP), a well-known restriction of Boolean Optimization. Binate Covering finds application in many areas of Computer Science and Engineering. In Artificial Intelligence, BCP can be used for computing minimum-size prime implicants of Boolean functions, of interest in Automated Reasoning and Non-Monotonic Reasoning. Moreover, Binate Covering is an essential modeling tool in Electronic Design Automation. The objectives of the paper are to briefly review branch-and-bound algorithms for BCP, to describe how to apply backtrack search pruning techniques from the Boolean Satisfiability (SAT) domain to BCP, and to illustrate how to strengthen those pruning techniques by exploiting the actual formulation of BCP. Experimental results, obtained on representative instances indicate that the proposed techniques provide significant performance gains for a large number of problem instances.

Keywords: binate covering problem, propositional satisfiability, branch-and-bound, backtrack search, non-chronological backtracking

1. Introduction

The generic Boolean Optimization problem as well as several of its restrictions are well-known computationally hard problems, widely used as modeling tools in Computer Science and Engineering. These problems have been the subject of extensive research work in the past (see, for example, [1]). In this paper we address the Binate Covering Problem (BCP), one of the restrictions of Boolean Optimization. BCP can be formulated as the problem of finding a satisfying assignment for a given Conjunctive Normal Form (CNF) formula subject to minimizing a given cost function. As with generic Boolean Optimization, BCP also finds many applications, including the computation of minimum-size prime implicants, of interest in Automated Reasoning and Non-Monotonic Reasoning [18], and as a modeling tool in Electronic Design Automation (EDA) [4,20].

In recent years, several powerful search pruning techniques have been proposed for solving BCP, allowing dramatic improvements in the ability to solving large and complex instances of BCP. (Details of the work on BCP can be found in [4,13,20].) Despite these improvements, and as with other NP-hard problems, additional search pruning ability

allows in general very significant gains, both in the amount of search and in the run times. The ultimate consequence of proposing new pruning techniques is the potential ability for solving new classes of instances.

The main objective of this paper is to propose additional techniques for pruning the amount of search in branch-and-bound algorithms for solving binate covering problems. These techniques correspond to generalizations and extensions of similar techniques proposed in the Boolean Satisfiability (SAT) domain, where they have been shown to be highly effective [2,15,22]. In particular, and to our best knowledge, we provide for the first time conditions which enable branch-and-bound algorithms to backtrack *non-chronologically* whenever bounding due to the cost function is required to take place. Although our main focus is on one particular bounding mechanism (maximum independent set of clauses), we also establish conditions for non-chronological backtracking with other bounding procedures.

The paper is organized as follows. In section 2 the notation used throughout the paper is introduced. Afterwards, branch-and-bound covering algorithms are briefly reviewed, giving emphasis to solutions based on SAT algorithms and in section 4 different bounding procedures are also described. In subsequent sections, we propose new techniques for reducing the amount of search. In particular we show how effective search pruning techniques from the SAT domain can be generalized and extended to the BCP domain. Experimental results are presented in section 8, and the paper concludes in section 9.

2.   Preliminaries

An instance C of a covering problem is defined as follows,

$$
\begin{aligned}
&\text{minimize} \sum_{j=1}^{n} c_j \cdot x_j, \\
&\text{subject to} \quad A \cdot x \geqslant b, \quad x \in \{0, 1\}^n,
\end{aligned}
\tag{1}
$$

where c_j is a non-negative integer cost associated with variable $x_j, 1 \leqslant j \leqslant n$ and $A \cdot x \geqslant b, x \in \{0, 1\}^n$ denote the set of m linear constraints. If every entry in the $(m \times n)$ matrix A is in the set $\{0, 1\}$ and $b_i = 1, 1 \leqslant i \leqslant m$, then C is an instance of the *unate covering problem* (UCP). Moreover, if the entries a_{ij} of A belong to $\{-1, 0, 1\}$ and $b_i = 1 - |\{a_{ij} : a_{ij} = -1, 1 \leqslant j \leqslant n\}|$, then C is an instance of the *binate covering problem* (BCP). Observe that if C is an instance of the binate covering problem, then each constraint can be interpreted as a propositional clause.

Conjunctive Normal Form (CNF) formulas are introduced next. The use of CNF formulas is justified by noting that the set of constraints of an instance C of BCP is equivalent to a CNF formula, and because some of the search pruning techniques described in the remainder of the paper are easier to convey in this alternative representation.

A propositional formula φ in *Conjunctive Normal Form* (CNF) denotes a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$. The formula φ consists of a conjunction of propositional

clauses, where each clause ω is a disjunction of literals, and a literal l is either a variable x_j or its complement $\bar{x}_j$. If a literal assumes value 1, then the clause is *satisfied*. If all literals of a clause assume value 0, the clause is *unsatisfied*. Clauses with only one unassigned literal are referred to as *unit*. Finally, clauses with more than one unassigned literal are said to be *unresolved*. In a search procedure, a *conflict* is said to be identified when at least one clause is unsatisfied. In addition, observe that a clause $\omega = (l_1 + \cdots + l_k)$, $k \leqslant n$, can be interpreted as a linear inequality $l_1 + \cdots + l_k \geqslant 1$, and the complement of a variable x_j, $\bar{x}_j$, can be represented by $1 - x_j$.

When a clause is unit (with only one unassigned literal) an assignment can be implied. For example, consider a propositional formula φ which contains clause $\omega = (x_1 + \bar{x}_2)$ and assume that $x_2 = 1$. For φ to be satisfied, x_1 must be assigned value 1 due to ω. Therefore, we say that $x_2 = 1$ *implies* $x_1 = 1$ due to ω or that clause ω *explains* the assignment $x_1 = 1$. These logical implications correspond to the application of the unit clause rule [7] and the process of repeatedly applying this rule is called *Boolean constraint propagation* [15,22]. It should be noted that throughout the remainder of this paper some familiarity with backtrack search SAT algorithms is assumed. The interested reader is referred to the bibliography (see, for example, [1,15] for additional references).

Covering problems are often solved by branch-and-bound algorithms [5,13,20]. In these cases, each node of the search tree corresponds to a selected unassigned variable and the two branches out of the node represent the assignment of 1 and 0 to that variable. These variables are named *decision variables*. The first node is called the *root* (or the top node) of the search tree and corresponds to the *first decision level*. The decision level of each decision is defined as one plus the decision level of the previous decision.

3. Search algorithms for covering problems

The most widely known approach for solving covering problems is the classical branch-and-bound procedure [20], in which *upper bounds* on the value of the cost function are identified for each solution to the constraints, and *lower bounds* on the value of the cost function are estimated considering the current set of variable assignments. The search can be pruned whenever the lower bound estimate is higher than or equal to the most recently computed upper bound. In these cases we can guarantee that a better solution cannot be found with the current variable assignments and therefore the search can be pruned. The algorithms described in [5,13,20] follow this approach.

Several lower bound estimation procedures can be used, namely the ones based on linear-programming relaxations [13], Lagrangian relaxations [17] or the Log-approximation approach [4]. Nevertheless, and for BCP, the approximation of a maximum independent set of clauses [4] is the most commonly used. The tightness of the lower bounding procedure is crucial for the algorithm's efficiency, because with higher estimates of the lower bound, the search can be pruned earlier. For a better understanding of lower bounding mechanisms, different methods will be described. We will address linear programming relaxations, the Log-approximation approach, and will emphasize the approximation of the maximum independent set of clauses. Covering algorithms also

incorporate several powerful reduction techniques, a comprehensive overview of which can be found in [4,20].

With respect to the application of SAT to Boolean Optimization, P. Barth [1] first proposed a SAT-based approach for solving pseudo-Boolean optimization (i.e. a generalization of BCP). This approach consists of performing a linear search on the possible values of the cost function, starting from the highest, at each step requiring the next computed solution to have a cost lower than the most recently computed upper bound. Whenever a new solution is found which satisfies all the constraints, the value of the cost function is recorded as the current lowest computed upper bound. If the resulting instance of SAT is not satisfiable, then the solution to the instance of BCP is given by the last recorded solution.

Additional SAT-based BCP algorithms have been proposed. In [14] a different algorithmic organization is described, consisting in the integration of several features from SAT algorithms in a branch-and-bound procedure, *bsolo*, to solve the binate covering problem. The *bsolo* algorithm incorporates the most significant features from both approaches, namely the bounding procedure and the reduction techniques from branch-and-bound algorithms, and the search pruning techniques from SAT algorithms.

The algorithm presented in [14] already incorporates the main pruning techniques of the GRASP SAT algorithm [15]. Hence, *bsolo* is a branch-and-bound algorithm for solving BCP that implements a non-chronological backtracking search strategy, clause recording and identification of necessary assignments. Mainly due to an effective conflict analysis procedure which allows non-chronological backtracking steps to be identified, *bsolo* performs better than other branch-and-bound algorithms in several classes of instances, as shown in [14]. However, non-chronological backtracking is limited to one specific type of conflict. In section 5 we describe how to apply non-chronological backtracking to *all* types of conflicts when using the approximation of a maximum independent set of clauses. Moreover, in section 7 we also describe how to apply the same concepts when using other lower bound estimation methods.

The main steps of a simplified version of the *bsolo* algorithm (see figure 1) can be described as follows:

1. Initialize the upper bound to the highest possible value as defined (i.e. given by $ub = \sum_{j=1}^{n} c_j + 1$).

2. The function *consistent_state* starts by checking whether the current state yields a conflict. This is done by applying Boolean constraint propagation and, in case a conflict is reached, by invoking the conflict analysis procedure, recording relevant clauses and proceeding with the search procedure or backtrack if necessary.

3. If a solution to the constraints has been identified, update the upper bound according to $ub = \sum_{j=1}^{n} c_j \cdot x_j$. (Observe that the only way to reduce the value of the current solution is to backtrack with the objective of finding a solution with a lower cost.)

4. Estimate a lower bound given the current variable assignments. If this value is higher than or equal to the current upper bound, issue a bound conflict and bound

```
int bsolo(φ) {
  ub = ∑ cj + 1;
  while (TRUE) {
    decide();
    if (!consistent_state())
      return ub;
    while (Estimate_LB() ⩾ ub) {
      Issue_LB_based_conflict();
      if (!consistent_state())
        return ub;
    }
  }
}
int consistent_state() {
  while (Deduce() == CONFLICT)
    if (Diagnose() == CONFLICT)
      return FALSE;
  if (Solution_found())
    Update_ub();
  return TRUE;
}
```

Figure 1. SAT-based branch-and-bound algorithm.

the search by applying the conflict analysis procedure to determine which decision node to backtrack to (using function *consistent_state*). Continue from step 2.

3.1. Bound conflicts

In *bsolo* two types of conflicts can be identified: *logical conflicts* that occur when at least one of the problem instance constraints becomes unsatisfied, and *bound conflicts* that occur when the lower bound is higher than or equal to the upper bound. When logical conflicts occur, the conflict analysis procedure from GRASP is applied and determines to which decision level the search should backtrack to (possibly in a non-chronological manner).

However, the other type of conflict is handled differently. In *bsolo*, whenever a bound conflict is identified, a new clause *must* be added to the problem instance in order for a logical conflict to be issued and, consequently, to bound the search. This requirement is inherited from the GRASP SAT algorithm where, for guaranteeing completeness, both conflicts and implied variable assignments *must* be explained in terms of the existing variable assignments [15]. With respect to conflicts, each recorded conflict clause is built using the assignments that are deemed responsible for the conflict to occur. If the assignment $x_j = 1$ (or $x_j = 0$) is considered responsible, the literal $\bar{x}_j$ (respectively, literal x_j) is added to the conflict clause. This literal basically states that in order to avoid the conflict one possibility is certainly to have instead the assignment $x_j = 0$ (respectively, $x_j = 1$). Clearly, by construction, after the clause is built its state is

unsatisfied. Consequently, the conflict analysis procedure has to be called to determine to which decision level the algorithm must backtrack to. Hence the search is bound.

Whenever a bound conflict is identified, one possible approach to building a clause to bound the search would be to include *all* decision variables in the search tree. In this case, the conflict would always depend on the last decision variable. Therefore, backtracking due to bound conflicts would necessarily be chronological (i.e. to the previous decision level), hence guaranteeing that the algorithm would be complete. Suppose that the set $\{x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1\}$ corresponds to all the search tree decision assignments and ω_{bc} is the clause to be added due to a bound conflict. Then we would have $\omega_{bc} = (\bar{x}_1 + x_2 + x_3 + \bar{x}_4)$. Again, the drawback of this approach (which was used in [14]) is that backtracking due to bound conflicts is always chronological, since it depends on all decision assignments made. In section 5 we propose a new procedure to build these clauses, which enables non-chronological backtracking due to bound conflicts.

4.   Computation of lower bounds

The estimation of lower bounds on the value of the cost function is a very effective method to prune the search tree and the accuracy of lower bounding procedures is critical for identifying areas of the search space where solutions to the constraints with lower values of the cost function cannot be found. In this section we review methods commonly used to estimate a lower bound on the value of the cost function in instances of BCP.

4.1. Maximum independent set of clauses

The maximum independent set of clauses (MIS) is a greedy method to estimate a lower bound on the value of the cost function based on an independent set of clauses. (A more detailed definition can be found, for example, in [4]).

The greedy procedure consists of finding a set *MIS* of disjoint unate clauses, i.e. clauses with only positive literals and with no literals in common among them. Since maximizing the cost of *MIS* is an NP-hard problem, a greedy computation is used, as shown in figure 2. The effectiveness of this method largely depends on the clauses included in *MIS*. Usually, one chooses the clause which maximizes the ratio between its weight and its number of elements.

The minimum cost for satisfying *MIS* is a *lower bound* on the solution of the problem instance and is given by

$$Cost(MIS) = \sum_{\omega \in MIS} Weight(\omega), \tag{2}$$

where

$$Weight(\omega) = \min_{x_j \in \omega} c_j. \tag{3}$$

```
maximal_independent_set(φ) {
  MIS = ∅;
  do{
    ω = choose_clause(φ);
    MIS = MIS ∪ {ω};
    φ = delete_intersecting_clauses(φ,ω);
  } while (φ ≠ ∅);
  return MIS;
}
```

Figure 2. Algorithm for computing a MIS.

4.2. Linear programming relaxations

Linear programming relaxations have long been used as lower bound estimation procedures in branch-and-bound algorithms for solving integer programming problems [17]. For the binate covering problem, the utilization of linear programming relaxations as a lower bound estimation method is proposed in [13]. Moreover, it is also claimed that in most cases the linear programming relaxation (LPR) bound is higher than the one obtained with the MIS approach.

The general formulation of the LPR for a covering problem is obtained from (1) as follows:

$$
\begin{aligned}
\text{minimize} \quad & z_{\text{lpr}} = \sum_{j=1}^{n} c_j \cdot x_j, \\
\text{subject to} \quad & A \cdot x \geqslant b, \\
& x \geqslant 0.
\end{aligned}
\tag{4}
$$

For simplicity the constraints $x \leqslant 1$ are not included. The solution of (1) is referred to as z_{cp}^*, whereas the solution of (4) is referred to as z_{lpr}^*.

It is well known that the solution z_{lpr}^* of (4) is a lower bound on the solution z_{cp}^* of (1) [17]. Basically, any solution of (1) is also a feasible solution of (4), but the converse is not true. Moreover, for a given solution of (4) where $x \in \{0, 1\}^n$, we necessarily have $z_{\text{cp}}^* = z_{\text{lpr}}^*$. Hence, the result follows. Furthermore, different linear programming algorithms can be used for solving (4), some of which with guaranteed worst-case polynomial run time [17].

4.3. Log-approximation

It is well known that the MIS approach can be very far from the minimum cost solution in specific cases of problem instance matrixes [5]. Given that the approximation provided by the greedy algorithm is of poor quality, tighter lower bounds can be established. In [5] a new lower bound computation algorithm for unate covering is introduced which guarantees a logarithmic ratio bound on the minimum cost solution.

The algorithm in figure 3 describes a procedure for constructing a greedy solution for a covering problem with a set φ of constraints to satisfy. At each step a decision

```
greedy_solution(φ) {
  SOL = ∅;
  while (φ ≠ ∅) {
```
$$var \ = \ min_{var \in Var(\varphi)} \ \frac{Cost(var)}{\Gamma(cov\_clauses(var, \varphi))};$$
```
    φ = φ − cov_clauses(var, φ);
    SOL = SOL ∪ var;
  }
  return SOL;
}
```

Figure 3. Algorithm for computation of a greedy solution.

variable *var* is chosen, the clauses that become satisfied by *var* are removed from φ and a solution set is updated. Observe that the algorithm was conceived to tackle unate covering problems [5], but it can be easily modified in order to attempt to find greedy solutions for binate covering instances.[1] The variable to add to the solution set is the one which minimizes the relation between its cost and the value given by Γ. Let γ be a positive weighting function defined on a set of clauses (e.g., the number of free literals). We can define Γ as:

$$\Gamma(\varphi') = \sum_{\omega \in \varphi'} \gamma(\omega). \tag{5}$$

It can be shown [5] that based on the greedy solution given by the algorithm in figure 3, it is possible to obtain a lower bound on the covering problem which is log-approximable to the optimum value z_{cp}^*. This result is also valid for binate covering whenever the greedy algorithm is able to find a feasible solution. The lower bound is given by:

$$\frac{Cost(\text{SOL})}{r}, \tag{6}$$

where

$$r = \sum_{k=1}^{\max_{\varphi'} \Gamma(\varphi')} 1/k. \tag{7}$$

5.   SAT-based pruning techniques for BCP

One of the main features of *bsolo* is the ability to backtrack non-chronologically when conflicts occur. This feature is enabled by the conflict analysis procedure inherited from the GRASP SAT algorithm. However, as illustrated in section 3.1, in the original *bsolo* algorithm non-chronological backtracking was only possible for logical conflicts. In the case of a bound conflict all the search tree decision assignments were used to

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[1] In binate covering, the greedy algorithm is unable to guarantee that a solution is found.

explain the conflict. Therefore, these conflicts would always depend on the last decision level and backtracking would necessarily be chronological.

In this section we describe how to compute sets of assignments that explain bound conflicts. Moreover, we show that these assignments are not in general associated with all decision levels in the search tree; hence non-chronological backtracking can take place.

A bound conflict in an instance of the binate covering problem (BCP) C arises when the lower bound is equal to or higher than the upper bound. This condition can be written as $C.path + C.lower \geqslant C.upper$, where $C.path$ is the cost of the assignments already made, $C.lower$ is a lower bound estimate on the cost of satisfying the clauses not yet satisfied (as given for example by an independent set of clauses), and $C.upper$ is the best solution found so far. From the previous equation, we can readily conclude that $C.path$ and $C.lower$ are the unique components involved in each bound conflict. (Notice that $C.upper$ is just the lowest value of the cost function for the solutions of the constraints computed earlier in the search process.) Therefore, we will analyze both $C.path$ and $C.lower$ components in order to establish the assignments responsible for a given bound conflict.

We start by studying $C.path$. Clearly, the variable assignments that cause the value of $C.path$ to grow are solely those assignments with a value of 1. Hence, we can define a set of literals ω_{cp}, such that each variable in ω_{cp} has positive cost and is assigned value 1:

$$\omega_{\mathrm{cp}} = \left\{ l = \bar{x}_j \colon Cost(x_j) > 0 \land x_j = 1 \right\} \tag{8}$$

which basically states that to decrease the value of the cost function (i.e. $C.path$) at least one variable that is assigned value 1 has instead to be assigned value 0.

We now consider $C.lower$. Let MIS be the independent set of clauses, obtained by the method described in section 4.1, that determines the value of $C.lower$. Observe that each clause in MIS is part of MIS because it is neither satisfied nor has common literals with any other clause in MIS. Clearly, for each clause $\omega_i \in MIS$ these conditions only hold due to the literals in ω_i that are assigned value 0. If any of these literals was assigned value 1, ω_i would certainly not be in MIS since it would be a satisfied clause. Consequently, we can define a set of literals that explain the value of $C.lower$:

$$\omega_{\mathrm{cl}} = \{l \colon l = 0 \land l \in \omega_i \land \omega_i \in MIS\}. \tag{9}$$

Now, as stated above, a bound conflict is solely due to the two components $C.path$ and $C.lower$. Hence, this bound conflict will hold as long as the following clause ω_{bc} is unsatisfied:

$$\omega_{\mathrm{bc}} = \omega_{\mathrm{cp}} \cup \omega_{\mathrm{cl}}. \tag{10}$$

(Observe that the set union symbol in the previous equation denotes a disjunction of literals.) As long as this clause is unsatisfied, the values of $C.path$ and $C.lower$ will remain unchanged, and so the bound conflict will exist. We can thus use this unsatisfied clause ω_{bc} to analyze the bound conflict and decide where to backtrack to, using the conflict analysis procedure of GRASP [15]. We should observe that backtracking can

be non-chronological, because clause ω_{bc} does not necessarily depend on all decision assignments. Moreover, due to the clause recording mechanism, ω_{bc} can be used later in the search process to prune the search tree. If these clauses would depend on all decision assignments, clause recording would not be used since the same set of decision assignments is never repeated in the search process.

Bound conflicts arise during the search process whenever we have $C.path + C.lower \geqslant C.upper$. Notice that when a new solution is found, $C.lower = 0$ because the independent set is empty (all clauses are satisfied) and $C.path$ is equal to the cost of the new upper bound. Therefore, when we update $C.upper$ with the new value, we have $C.path + C.lower = C.upper$ and a bound conflict is issued in order to backtrack in the search tree. These bound conflicts are just a particular case and the same process we described in this section is applied in order to build the conflict clause.

6.   Reducing dependencies in bound conflicts

As shown in the previous section, in branch-and-bound BCP algorithms it is possible to establish conditions for implementing non-chronological backtracking due to bound conflicts. However, the ability to backtrack non-chronologically is strongly related with the ability for identifying a small set of assignments that explain each bound conflict. Sets of assignments that include many assignments irrelevant for actually explaining the bound conflict can drastically reduce the ability to backtrack non-chronologically. Hence, after computing explanations for bound conflicts, using the techniques described in the previous section, the next step is to identify assignments that can be discarded from each explanation by proving them irrelevant for the bound conflict to take place.

In this section we propose different techniques for reducing dependencies in the explanations of bound conflicts, hence reducing the number of literals in ω_{bc}.

6.1. Relating C.path and C.lower

Let l_j be a literal such that $l_j \in \omega_{cp}$ and $l_j \notin \omega_{cl}$. Then l_j is in ω_{bc} only due to the $C.path$ component explaining the bound conflict. Let MIS be the independent set, computed with the procedure described in figure 2, which is used to obtain the value of $C.lower$. In this situation, literal l_j can be removed from ω_{cp} provided the following conditions apply:

– There exists a satisfied clause ω_i such that $\bar{l}_j$ is the only literal which currently satisfies ω_i.
– All literals of ω_i besides l_j must be positive, unassigned and must not intersect MIS (so that ω_i can be added to MIS if l_j assumes value 0).
– All literals in ω_i must have a cost higher than or equal to the cost of literal l_j.
– No clause in MIS contains l_j.

This reduction step can be made because if $l_j = 0$, ω_i would be in the independent set and the lower bound value would not decrease. Therefore, literal l_j can be deemed irrelevant for explaining the bound conflict and can be removed from ω_{bc}.

As an example, let us suppose that variables x_1, x_2 and x_3 belong to the cost function with the same cost and $x_1 = 1$. If a bound conflict occurs, from (8) $\bar{x}_1$ would be in ω_{bc}. However, suppose that clause $\omega_i = (x_1 + x_2 + x_3)$ is satisfied only due to x_1, i.e., x_2 and x_3 are unassigned. If x_2 and x_3 do not belong to any clause in *MIS*, $\bar{x}_1$ can be removed from ω_{bc} because $x_1 = 1$ is not relevant for the conflict. If variable x_1 was unassigned or assigned value 0, ω_i would be in *MIS* and the bound conflict would still occur.

It is interesting to observe that we can generalize the second condition, allowing ω_i to have positive literals whose variables are assigned value 0. Let us consider the example clause $\omega_i = (x_1 + x_2 + x_3 + x_4)$. Let $x_1 = 1$ and $x_2 = 0$. Moreover, let the cost of x_1 be no greater than the cost of x_2, let x_3, x_4 be such that ω_i would be in *MIS* if $x_1 = 0$, and let no other clause in MIS contain literal x_2. In this situation, the dependency on x_1 can be removed, and the dependency on x_2 need not be considered. Indeed, with $x_1 = 0$, ω_i would be in *MIS* and so the cost would not decrease. In addition, since the cost of x_2 is larger than or equal to the cost of x_1, by assigning value 1 to x_2, the cost would also not decrease. Hence the result follows. One should note that the same reasoning applies for an *arbitrary* number of variables assigned value 0 in a given clause with a single literal assigned value 1.

Next we show how ω_{cl} can be simplified by evaluating the consequences of modifying the value of some literals on the value of *C.path*.

Suppose we have a literal $l = x_j$, with $l \in \omega_{cl}$ and let $x_j = 0$. If x_j only belongs to one clause ω_i of the independent set and its cost is greater than or equal to the minimum cost of ω_i, then l can be removed from ω_{bc}. To better understand how this is possible, suppose instead that $x_j = 1$. In this situation, ω_i would not be in the independent set (it would be a satisfied clause) and the *C.lower* component would be lower.[2] However, since the cost of the variable is higher than or equal to the minimum cost of ω_i, the *C.path* component would be higher, and hence the conflict would still hold. So, the assignment $x_j = 0$ is irrelevant for the conflict to arise and literal l can be removed from ω_{bc}. Observe that the same reasoning still applies even if a clause ω_i, containing literal $x_j = 0$, contains any number of other literals assigned value 0.

Another reduction technique consists of using a satisfied clause to reduce a dependency from ω_{cl}. Let us consider the following set of clauses,

$$
\begin{aligned}
\omega_1 &= (x_1 + x_2 + x_3), \\
\omega_2 &= (x_1 + x_4 + x_5), \\
\omega_3 &= (\overline{x_1} + x_3 + x_4)
\end{aligned}
\tag{11}
$$

---

[2] In fact, if the *C.lower* would be recomputed all over again, it is not guaranteed that it would decrease. Nevertheless, we know that without clause ω_i satisfied by $x_j = 1$, $MIS \backslash \{\omega_i\}$ it is still an independent set of clauses. Therefore, $MIS \backslash \{\omega_i\}$ can be used as a *low* estimate of *C.lower*.

with $x_1 = 0$, x_2, x_3, x_4, x_5 unassigned, and let ω_1 and ω_2 be part of *MIS*. Let the cost of x_2, x_3, x_4, x_5 be less than or equal to the cost of x_1. Finally, let no other clause in *MIS* contain x_1. If x_1 would be assigned value 1, *C.lower* would decrease by 1 since ω_1 and ω_2 would be satisfied, but ω_3 would now be in *MIS*. However, *C.path* would be raised due to the cost of x_1 and the conflict would still hold. Hence, the dependency on x_1 can be removed.

6.2. Using excess cost value

Let us consider a bound conflict and let $diff = (C.path + C.lower) - C.upper$. Clearly, $diff \geqslant 0$.

It is plain that if *C.path* was lower by *diff*, the bound conflict would still hold since we would then have $C.path + C.lower = C.upper$. Therefore, we may conclude that not all assignments in *C.path* are necessary for explaining the conflict, since if some assignments were not made, we would still have a bound conflict. In this case, it is possible to remove some literals from ω_{cp} as long as their cost is lower than or equal to *diff*.

Moreover, the value of *diff* can also be used for reducing dependencies from *C.lower*. Notice that if we remove a subset of clauses *D_MIS* from *MIS* (used to obtain *C.lower*) such that,

$$Cost(D\_MIS) \leqslant diff, \tag{12}$$

where

$$Cost(D\_MIS) = \sum_{\omega \in D\_MIS} Weight(\omega), \tag{13}$$

then the lower bound conflict will still hold since $C.path + C.lower \geqslant C.upper$, where *C.lower* is now obtained from the independent set of clauses $MIS \setminus D\_MIS$. Therefore, the lower bound conflict clause ω_{bc} can still be built using (10), but ω_{cl} can now be reformulated as

$$\omega_{cl} = \{l: l = 0 \wedge l \in \omega_i \wedge \omega_i \in MIS \setminus D\_MIS\}. \tag{14}$$

Moreover, the simplifications described above for ω_{cl} can now be applied to the resulting ω_{cl}.

One should note that the reduction of the number of dependencies relies on which clauses we choose to include in *D_MIS*. If a clause from *MIS* is selected with assigned literals belonging to ω_{bc} because of other clauses in *MIS* or due to ω_{cp}, then the dependencies are exactly the same. Therefore, it is desirable that *D_MIS* be a subset of *MIS* such that the number of dependencies in ω_{bc} be minimum. Currently, in *bsolo*, a greedy procedure is used for selecting the clauses to remove from *MIS*.

6.3. Resolution-induced dependency reduction

In this section we illustrate how the resolution operation [19] can be used for establishing conditions that permit the elimination of dependencies. We should note that the proposed conditions, even though based on the resolution operation, do not require the explicit creation of new clauses.

The conditions proposed subsequently can be applied to removing dependencies from ω_{cp} and ω_{cl}. In all cases, we use examples to illustrate the application of resolution, but provide the necessary conditions for generic application.

We start by studying simplifications to ω_{cp} established with the resolution operation. Let us consider the following set of clauses,

$$\begin{aligned}
\omega_1 &= (x_1 + x_2 + x_3), \\
\omega_2 &= (\overline{x_1} + x_2 + x_4)
\end{aligned} \tag{15}$$

with $x_2 = 1$, and such that x_3, x_4 are not covered by the currently computed *MIS*. x_1 can either be assigned or unassigned, and can either be or not be covered by the currently computed *MIS*. By applying resolution between ω_1 and ω_2, with respect to x_1, we obtain the resulting clause $\omega_3 = c(\omega_1, \omega_2, x_1) = (x_2 + x_3 + x_4)$. Now, ω_3 is certainly satisfied solely by x_2. Hence, we can conclude that the dependency on x_2 can be removed by applying the results of sections 6.1 and 6.2 on simplifying ω_{cp}. Notice that x_1 can be *any* variable. However, if x_1 is unassigned and not covered by *MIS*, then we can immediately apply the previous results on simplifying ω_{cp}.

Next, we illustrate one additional form of using the resolution operation for removing dependencies. As an example, assume a bound conflict, and consider the following set of clauses,

$$\begin{aligned}
\omega_1 &= (x_1 + x_2 + x_3), \\
\omega_2 &= (\overline{x_1} + x_4 + x_5),
\end{aligned} \tag{16}$$

where x_1 is assigned either value 0 or 1, its cost is 0, and such that the dependency on x_1 is only due to ω_1 or ω_2. Furthermore, let us assume that ω_1 would be part of *MIS* with $x_1 = 0$, and that ω_2 would be part of *MIS* with $x_1 = 1$. In this situation the dependency on x_1 can be removed. Notice that if the cost of x_1 is non-zero, then the removal of the dependency on x_1 is guaranteed by the previous results (section 6.1) on simplifying ω_{cl}.

Clearly, the application of the resolution operation can be generalized and used for eliminating more than one variable, the only drawback being the computational effort involved.

7. Bound pruning with other lower bound methods

Sections 5 and 6 describe how to incorporate non-chronological backtracking whenever a bound conflict occurs, either when a new solution is found or when the computed lower bound estimate is higher than or equal to the best solution found so far.

Notice that in the first case, the procedure does not depend on the method used for computing the lower bound,[3] whereas in the second case it is assumed that the lower bound computation method is the maximum independent set (MIS) described in section 4.1. If a different lower bound estimation procedure is used, the techniques proposed so far to allow non-chronological backtracking cannot be utilized, since a different procedure will be required for identifying an explanation for the bound conflict. In the remainder of this section we present a theoretical framework which also allows non-chronological backtracking when the other lower bounding procedures described in section 4 are used.

7.1. Pruning with LP-relaxation lower bounds

Linear programming relaxations (LPR) are a powerful method to estimate a lower bound value for binate covering instances [13]. However, the resulting backtrack from a bound conflict when using LPR has always been chronological. As mentioned previously, the techniques presented in sections 5 and 6 are useless since the application of the MIS procedure is assumed. The naive approach to build a clause to bound the search in bound conflicts when using LPR would be to include all decision variables in the search tree. However, as stated in section 3.1, the resulting backtrack would necessarily be chronological. In this section, we present a new framework that allows non-chronological backtracking in bound conflicts when linear programming relaxations are used to estimate the lower bound value.

Remember that a bound conflict occurs when $C.path + C.lower \geqslant C.upper$, in which case a set of assignments that explains the conflict must be identified. Therefore, we must identify the assignments that explain the value on $C.path$ (ω_{cp}) and $C.lower$ (ω_{cl}) in order to build the bound conflict clause ω_{bc} to bound the search. Notice that the value on $C.path$ is independent on the lower bound computation procedure and it can be build as described in section 5. However, not every reduction technique presented in section 6 can be applied to ω_{cp}, since some of these techniques depend on the fact that the MIS procedure is used to compute $C.lower$.

The approach to build ω_{cl} must be different from what was presented in section 5, since $C.lower$ depends on the value given by the LP-solver. Therefore, the information provided by the LP-solver must be used in order to implement a non-chronological backtracking search strategy in a lower bound conflict when using LPR for bound computation.

Given the value of $C.lower$ using LPR as formulated in section 4.2, let S be the set of constraints with *slack* [4] variables assigned value 0. Observe that these are the constraints which actually limit the value of $C.lower$, and so will be referred to as the *active* constraints. When using LP-relaxations to obtain the value of $C.lower$, the literals that assume value 0 in the active constraints are directly responsible for its value. These liter-

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[3] When a new solution is found, all clauses are satisfied and the lower bound procedure used in the algorithm is irrelevant.

[4] See [3] for a definition of slack and artificial variables.

als correspond to the set ω_{cl} in the bound conflict clause. Applying this reasoning to both assignments of a given variable, allows implementing non-chronological backtracking.

We now illustrate how non-chronological backtrack is possible when using LPR. Let $C.lower$ be computed using LP-relaxations and let d_{LPR} denote the highest decision level, besides the current decision level, of the zero-valued literal assignments in the constraints for which the slack variables assume value 0. Let the current decision level be $d + k$ and let d be the lowest decision level such that,

$$C.path(d) + C_l.lower(d + k) \geqslant C.upper, \tag{17}$$

$$C.path(d) + C_r.lower(d + k) \geqslant C.upper. \tag{18}$$

The highest decision levels involved, besides the current decision level $d + k$ are, respectively, $d_{LPR,l}$ from (17) and $d_{LPR,r}$ from (18). In this situation, the search process can backtrack to decision level $\max\{d, d_{LPR,l}, d_{LPR,r}\}$. Observe that the highest decision level in $\max\{d, d_{LPR,l}, d_{LPR,r}\}$ denotes the first possibility for one of the constraints in (17) or (18) not to hold. Hence, the search process can safely and non-chronologically backtrack to this decision level.

7.2. Pruning with log-approximation lower bounds

When $C.lower$ is estimated using the Log-approximation method described in section 4.3, its value depends on the variable assignments chosen by the greedy algorithm (see figure 3). Notice that each time a variable assignment is chosen, it depends on the value of function Γ. Therefore, choosing a variable assignment depends on the clauses which become satisfied with that assignment.

Suppose an assignment to variable x_j is chosen at iteration k of the greedy algorithm. This assignment is due to the fact that there is a set of clauses given by $cov\_clauses(x_j, \varphi(k)')$[5] which become satisfied and this set of clauses allows variable x_j to be chosen by the algorithm. Therefore, the clauses in $cov\_clauses$ provide an explanation for the assignment to variable x_j. Moreover, the literals assigned value 0 in $cov\_clauses(x_j, \varphi(k)')$ are the ones deemed responsible since if they were to have a different value, $cov\_clauses(x_j, \varphi(k)')$ would be a smaller set which could cause the assignment to variable x_j unrequired and hence $C.lower$ could be lower. Notice that if any of the literals considered responsible were to satisfy some of these clauses by having the opposite value, the set of clauses to satisfy (given by $cov\_clauses$) would be smaller and a lower value for $C.lower$ could be obtained, possibly solving the bound conflict situation.

Let $C.lower$ be estimated using the Log-approximation method and let SOL be the solution found by the greedy algorithm in n iterations which yields a bound conflict. In that case, a bound conflict clause ω_{bc} must be created to bound the search. The explanation on $C.path$ is determined as described previously in section 5, since it does not

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[5] Notice that $\varphi(k)'$ denotes the set of clauses still to satisfy at iteration k of the greedy algorithm.

depend on the lower bound estimation method. Moreover, the explanation on *C.lower* is given by:

$$\omega_{cl} = \{l\colon l = 0 \wedge l \in \omega_i \wedge \omega_i \in \pi(n)\}, \tag{19}$$

where $\pi(n)$ is the set of all clauses covered by the assigned variables chosen until iteration n which is equivalent to:

$$\pi(n) = cov\_clauses\big(SOL[1], \varphi(1)'\big) \cup \cdots \cup cov\_clauses\big(SOL[n], \varphi(n)'\big), \tag{20}$$

where $SOL[k]$ is the selected assignment at iteration k in the greedy algorithm.

Notice that at iteration n all clauses from φ (clauses that are not yet satisfied) are in $\pi(n)$, since all are covered at iteration n of the greedy algorithm. Nevertheless, it is possible that the resulting bound conflict clause ω_{bc} does not depend on the last decision assignment level and non-chronological backtracking can take place.

8.   Experimental results

In this section we compare different algorithms for solving BCP on example instances taken from digital circuit testing problems [9]. Due to space limitations, only the most representative instances are presented.

Table 1
Results for bsolo levels 0 and 1.

Benchmark	min.	Level 0				Level 1			
		CPU	Dec.	NCB	Jump	CPU	Dec.	NCB	Jump
c1908_F469@0	–	ub23	72211	18	6	ub13	117079	721	9
c1908_F953@0	4	438.56	2228	6	2	237.54	1394	61	10
c3540_F20@1	6	ub 6	10539	56	7	1045.14	3359	218	7
c432_F1gat@1	8	1414.04	15844	7	3	575.16	14756	608	53
c432_F37gat@1	9	ub15	143452	8	3	ub15	218136	35785	21
c499_Fic2@1	–	ub41	1000029	0	1	ub41	1003200	1586	3
c6288_F35gat@1	4	286.07	1255	0	1	107.69	756	41	42
c6288_F69gat@1	6	ub6	12379	7	15	1413.17	4048	110	41
9symml_F1@1	9	8.30	351	14	5	7.41	335	23	5
9symml_F6@0	9	6.91	301	13	4	6.05	272	23	4
alu4_Fj@0	6	249.89	1566	11	6	185.59	1292	55	4
alu4_Fl@1	6	159.31	1036	10	3	146.01	999	81	4
apex2_Fv14@1	10	20.48	974	0	1	20.15	908	48	4
apex2_Fv17@1	12	27.85	1163	5	4	23.38	1082	70	5
duke2_Fv5@1	5	36.88	592	6	3	26.05	515	52	9
duke2_Fv7@0	5	16.61	356	0	1	13.31	335	33	12
misex3_Fa@0	9	117.19	1526	9	4	56.78	898	83	14
misex3_Fb@1	8	98.25	1128	10	3	83.91	1038	71	8
spla_Fv10@0	7	42.31	809	7	3	34.78	766	104	7
spla_Fv14@0	8	55.00	1064	1	5	38.93	914	120	12

Table 2
Results for bsolo levels 2 and 3.

Benchmark	min.	Level 2				Level 3			
		CPU	Dec.	NCB	Jump	CPU	Dec.	NCB	Jump
c1908_F469@0	–	ub13	111277	1049	7	ub13	111386	1057	7
c1908_F953@0	4	241.04	1416	65	10	240.60	1416	65	10
c3540_F20@1	6	1009.86	3221	226	7	907.40	2939	213	7
c432_F1gat@1	8	540.20	14117	647	53	541.48	14117	647	53
c432_F37gat@1	9	ub14	286225	48534	21	ub14	286490	48534	21
c499_Fic2@1	–	ub41	1003200	1586	3	ub41	1003200	1586	3
c6288_F35gat@1	4	108.83	756	41	42	44.42	555	39	42
c6288_F69gat@1	6	970.29	3002	100	41	608.99	2198	94	41
9symml_F1@1	9	8.02	335	23	5	7.51	335	23	5
9symml_F6@0	9	6.52	272	23	4	6.12	272	23	4
alu4_Fj@0	6	157.07	1116	51	4	145.73	1034	46	5
alu4_Fl@1	6	145.02	1002	84	5	132.75	933	73	5
apex2_Fv14@1	10	20.21	904	55	4	20.41	936	60	4
apex2_Fv17@1	12	24.94	1089	68	5	23.60	1058	78	5
duke2_Fv5@1	5	24.89	495	49	9	26.60	495	49	9
duke2_Fv7@0	5	13.01	333	32	12	12.93	332	32	12
misex3_Fa@0	9	55.51	879	81	14	55.18	879	81	14
misex3_Fb@1	8	81.40	1006	69	8	80.47	1006	69	8
spla_Fv10@0	7	35.29	765	106	7	33.89	764	106	7
spla_Fv14@0	8	28.32	784	114	10	28.23	785	113	10

For the experimental results given below, the CPU times were obtained on a SUN Sparc Ultra I, running at 170 MHz, and with 100 MByte of physical memory. In all cases the maximum CPU time that each algorithm was allowed to spend on any given instance was 1 hour. When the algorithm was unable to solve the instance due to time restrictions, the best upper bound found at the time is shown. Otherwise, if no upper bound was computed, the reason of failure is shown, which was either due to the time (`time`) or memory (`mem.`) limits imposed. In tables 1 and 2, besides the time taken and the number of decisions made to solve the instances, it is also shown the number of non-chronological backtracks and the highest jump made in the search tree.

The experimental procedure consisted of running a selected set of problem instances with the *bsolo* algorithm, as described in sections 3, 5 and 6. These results are shown in tables 1 and 2. Here we can see the differences between several levels of computational effort in identifying dependencies in bound conflicts. Level 0 corresponds to section 3 where *bsolo* can only backtrack chronologically in bound conflicts, while level 1 corresponds to the identification of dependencies described in section 5. The techniques for reducing the number of dependencies presented in sections 6.1 and 6.2 are only considered into level 2. Level 3 differs from the previous level since it also includes the resolution-based dependency reduction techniques from section 6.3.

In table 1 we can clearly observe significant gains due to the fact that non-chronological backtracking in bound conflicts is implemented in level 1. In several cases we can observe the increase on both the number of non-chronological backtracks and

Table 3
Algorithm comparison.

Benchmark	min.	Algorithms			
		lp_solve	*scherzo*	*opbdp*	*bsolo*
c1908_F469@0	–	time	time	ub 24	ub13
c1908_F953@0	4	time	3424.81	ub 26	240.60
c3540_F20@1	6	time	mem.	ub 13	907.40
c432_F1gat@1	8	ub 15	time	1148.27	541.48
c432_F37gat@1	9	time	time	3574.44	ub14
c499_Fic2@1	–	time	time	ub 41	ub41
c5315_F43@0	3	2.6	0.92	30.38	0.67
c5315_F54@1	5	time	mem.	time	42.06
c6288_F35gat@1	4	time	mem.	1330.95	44.42
c6288_F69gat@1	6	time	mem.	ub 9	608.99
9symml_F1@1	9	ub 9	28.64	2.01	7.51
9symml_F6@0	9	ub 9	29.44	1.59	6.12
alu4_Fj@0	6	time	879.05	413.71	145.73
alu4_Fl@1	6	time	1638.98	557.14	132.75
apex2_Fv14@1	10	ub 10	mem.	624.07	20.41
apex2_Fv17@1	12	time	mem.	532.94	23.60
duke2_Fv5@1	5	time	mem.	82.01	26.60
duke2_Fv7@0	5	time	mem.	18.20	12.93
misex3_Fa@0	9	time	mem.	182.41	55.18
misex3_Fb@1	8	time	mem.	983.55	80.47
spla_Fv10@0	7	time	mem.	202.98	33.89
spla_Fv14@0	8	time	mem.	215.79	28.23

the highest jump in the search tree. For example, instance `c3540_F20@1` could not be solved with *bsolo*'s level 0, but was solved in less than one third of the given time limit with the identification of dependencies in bound conflicts.

Table 2 presents the results for levels 2 and 3. For each of these levels, additional gains are observed, mostly due to more non-chronological backtracks. With the application of techniques for reducing the number of dependencies, smaller set of assignments are declared responsible for the bound conflicts and more non-chronological backtracks are possible.

Finally, in table 3 we can observe the results of several other algorithms on the same set of instances. Clearly, *lp_solve* [16] (a generic Integer Linear Programming solver) is unable to solve almost all instances given the time limit. Notice that only in some cases was it able to find an upper bound to problem instances. *scherzo* [5], a state-of-the-art BCP solver, which incorporates several powerful pruning techniques in a classical branch-and-bound algorithm, is also unable to solve most of the example instances. The SAT-based linear search algorithm *opbdp* [1] is able to solve most instances, hence suggesting that these instances are well-suited for SAT-based solvers. Notice however that *bsolo* is faster than *opbdp* in most examples, and in some cases the improvement exceeds 1 order magnitude.

9. Conclusions

This paper extends well-known search pruning techniques, from the Boolean Satisfiability domain, to branch-and-bound algorithms for solving the Binate Covering Problem. The paper also describes conditions that allow for non-chronological backtracking in the presence of bound conflicts when using maximum independent sets as a lower bound mechanism. To our best knowledge, this is the first time that branch-and-bound algorithms are augmented with the ability for backtracking non-chronologically in the presence of conflicts that result from bound conditions. In addition, we have established conditions for reducing the size of bound conflict explanations, which further elicits non-chronological backtracking. Moreover, we also describe how to enable non-chronological backtracking with other bounding procedures, namely linear programming relaxations and Log-approximation [4].

Preliminary results obtained on several instances of the Binate Covering Problem indicate that the proposed techniques are indeed effective and can be significant for specific classes of instances, in particular for instances of covering problems with sets of constraints that are hard to satisfy.

Future research work will naturally include seeking further simplification of the clauses created for each type of conflict, and generalizing the *bsolo* algorithm to other Boolean optimization problems.

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